last time

K compl. disc. valued, L/K finite ext.

$\mathcal{O}_K \supseteq \mathfrak{m}_K \ni \pi_K \qquad \mathcal{O}_L \supseteq \mathfrak{m}_L \ni \pi_L$

$v_K : K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}, \quad v_L : L \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$

$v_K : \overline{K} \twoheadrightarrow \mathbb{Q} \cup \{\infty\}$ unique ext. of v_K to $\overline{K}$

$$v_K(\pi_L) = \frac{1}{e}, \quad e = e(L|K)$$

(Equiv: $\mathfrak{m}_K \cdot \mathcal{O}_L = \mathfrak{m}_L^e$)

$f = f(L|K) = [k_L : k]$, k_L, k res. field of $\mathcal{O}_L, \mathcal{O}_K$

Saw: $n := [L:K] = e \cdot f$

L/K totally ramified if $n = e$

In this case: The minimal poly. f of π_L over K has $\mathcal{O}_K$ and f is an

Eisenstein polynomial.

Moreover, $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$

Conversely: Assume that $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in \mathcal{O}_K[x]$

is Eisenstein, i.e.

$v_K(a_0) = 1$, $v_K(a_i) \geq 1 \ \forall i = 0, \dots, n-1$

Claim: $L := K[x]/f(x)$ is a totally ramified ext. of K of deg n, $\mathcal{O}_L = \mathcal{O}_K[\bar{x}]$, $\pi_L := \bar{x}$ is a uniformizer in $\mathcal{O}_L$

Proof of claim:

STS: $\mathcal{O}_L = \mathcal{O}_K[\bar{x}]$

„$\supseteq$" ✓ as $\bar{x} \in L$ is integral over $\mathcal{O}_K$ and $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L

„$\subseteq$" STP: $\mathcal{O}_K[\bar{x}]/\pi_K \twoheadrightarrow \mathcal{O}_L/\pi_K$

(lemma last time)

Now:

$$v_L(\bar{x}) = e(L|K) \cdot v_K(\bar{x})$$

$$\overset{\mathbb{Z}_{\geq 0}}{\rotatebox{90}{$\in$}} \quad = \frac{e(L|K)}{n} \cdot v_K(N_{L/K}(\bar{x}))$$

$$= \frac{e(L|K)}{n} \cdot v_K(a_0) = \underset{\leq 1}{\frac{e(L|K)}{n}} > 0$$

$\Rightarrow$ $v_L(\bar{x}) = 1$ & $n = e(L|K)$

i.e. L/K is totally ramified

& $\pi_L := \bar{x}$ is uniformizer in $\mathcal{O}_L$

Moreover, $k_L = k$ (as $f(L|K) = \frac{n}{e} = 1$)

Upshot:

$$\mathcal{O}_K[\pi_L]/\pi_K \cong \mathcal{O}_K[x]/(\pi_K, f(x)) \cong k[x]/x^n$$

& $\mathcal{O}_L/\pi_K \cong \mathcal{O}_L/\mathfrak{m}_L^n \cong k[z]/z^n$

z = res. class of π_L

& $\mathcal{O}_K[\pi_L]/\pi_K \xrightarrow{\sim} \mathcal{O}_L/\pi_K$

$\cong$

$k[x]/x^n \xrightarrow{\sim} k[z]/z^n$

$x \longmapsto z$

$\triangle$ Claim

Unramified extensions

Recall: L/K unramified if $n = [L:K] = f(L|K)$
& k_L is separable over k

(k_L sep. over k is always satisfied if k perfect, e.g. $K/\mathbb{Q}_p$ finite)

Fix K as before

$$F: \{\text{finite ext. of } K\} \to \{\text{finite ext. of } k\}$$
$$L \longmapsto k_L$$

Assume k'/k is finite separable

$\Rightarrow \exists\ K'/K$ finite with natural isom.

$$\mathrm{Hom}_{K\text{-alg}}(K', L) \simeq \mathrm{Hom}_{k\text{-alg}}(k', k_L)$$

Notation: $G(k') := K'$ (K' is unique up to can. isom.

Moreover, $G(k') = K'$ is unramified over K & $F \circ G(k') = k'$

Proof: Write $k' = k[x]/\bar{f}$ with $\bar{f} \in k[x]$ separable, monic

Let $f \in \mathcal{O}_K[x]$ be a monic lift of $\bar{f}$

$\Rightarrow f \in \mathcal{O}_K[x]$ is irreducible

Set $K' := K[x]/f(x)$

$\Rightarrow \mathrm{Hom}_{K\text{-alg}}(K', L)$

$$\cong \{ \alpha \in L \mid f(\alpha) = 0 \}$$

$$\overset{\uparrow}{\underset{f \in \mathcal{O}_K[x]}{\cong}} \{ \alpha \in \mathcal{O}_L \mid f(\alpha) = 0 \} \qquad [\mathcal{O}_L = \text{int. closure of } \mathcal{O}_K \text{ in } L)$$

$$\underset{\substack{\text{Hensel's La} \\ + \bar{f} \text{ sep.}}}{\cong} \{ \bar{\alpha} \in k_L \mid \bar{f}(\bar{\alpha}) = 0 \}$$

$$= \mathrm{Hom}_{k\text{-alg}}(k', k_L)$$

Note: $[K' : K] = [k' : k]$

& ex. emb. $k' \hookrightarrow k_{K'}$

(corresp. to $\mathrm{Id}_{K'} \in \mathrm{Hom}_{K\text{-alg}}(K', K)$

$$\cong$$

$$\mathrm{Hom}_{k\text{-alg}}(k', k_{K'})$$

$$\Rightarrow \quad k' \underset{\substack{\uparrow \\ \text{naturally}}}{\cong} k_{K'}$$

In part, K' unramified over K $\square$

Conclusion:

G induces an equivalence (with inverse F)

$$\{\text{finite, sep. ext of } k\} \xleftrightarrow{1:1} \{\text{finite, unram. ext. of } K\}$$

$$k' \longmapsto G(k')$$

$$k_L \longleftarrow L$$

In part, k'/k Galois if and only if $G(k')$ is Galois over K

Indeed, if L/K is finite (& arbitrary)

$\Rightarrow$ $\exists$ can. morph $G(\underbrace{k_{L,sep}}_{\substack{\text{largest sep. subext. of} \\ k \text{ in } k_L}}) \hookrightarrow L$

which is an isom. if and only if
L/K is unramified

$(L_0 := G(k_{L,sep})$ is called the
maximal unramified sub-ext. of $L/K)$

G, F extend to infinite extensions

$\left(L/K \text{ unramified if } L = \bigcup_{\substack{L_i/K \\ \text{finite + unram.}}} L_i \right)$

$\Rightarrow \{ \text{sep. (\& algebraic) ext. of } k \}$

$\simeq \{ \text{unram. ext. of } K \}$

If k^{sep}/k sep. closure

$\Rightarrow G(k^{sep}) =: K^{un}$ maximal unramified extension of K

$$G(k^{sep}) = \varinjlim_{\substack{k'/k \text{ finite, sep.} \\ k' \subseteq k^{sep}}} G(k')$$

K^{un}/K is Galois &

$$\mathrm{Gal}(K^{un}/K) \cong \mathrm{Gal}(k^{sep}/k)$$

canonically (+ homeo.)

Ex: $K = \mathbb{Q}_p \Rightarrow k = \mathbb{F}_p$ perfect

and $\bar{k} = \overline{\mathbb{F}_p} = \bigcup_{n \geq 1} \mathbb{F}_{p^n}$

$G(\mathbb{F}_{p^n}) \twoheadrightarrow G(\mathbb{F}_{p^m})$

$\mathbb{F}_{p^n}$ is the splitting field of $X^{p^n} - X$

$\Rightarrow \forall n \geq 1$ ex. (up to isom. unique)

ext. K_n (or $\mathbb{Q}_{p^n}$) of $\mathbb{Q}_p$ of degree n

which is unramified

Concretely. $\mathbb{Q}_{p^n}$ splitting field of $X^{p^n} - X$

$\Rightarrow \mathbb{Q}_p^{un} = \bigcup_{(m,p)=1} \mathbb{Q}_p(\zeta_m)$, $\mu_m = \{x \in \overline{\mathbb{Q}_p} \mid x^m = 1\}$

Exercise

(note $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ is <u>ramified</u>)

$$\overline{\mathbb{F}_p} = \varinjlim_{n} \mathbb{F}_{p^n}$$

← trs. maps are $\mathbb{F}_{p^m} \hookrightarrow \mathbb{F}_{p^n}$

$m \mid n$

$$\Rightarrow \mathbb{Q}_p^{un} = \varinjlim_{n} G(\mathbb{F}_{p^n})$$

← trs. maps are

$G(\mathbb{F}_{p^m}) \hookrightarrow G(\mathbb{F}_{p^n})$

As k' is only def. up to unique isom. how is $G(k')$ actually defined?

Answer: For two choices k', k''

let $\vartheta_{k',k''} : k' \to k''$ be the unique implicit isom.

Set $G(k') := \varprojlim_{k'} k' = \{ (x_{k'}) \in \prod k' \mid \vartheta_{k',k''}(x_{k'}) = x_{k''} \}$

← trs. maps given by $\vartheta_{k',k''}$

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p^{un}/\mathbb{Q}_p)$$

$$\cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$$

pro-solvable
(easier than $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$)

(LCFT: $\mathbb{Q}_p^{ab} \cong \bigcup_m \mathbb{Q}_p(\zeta_m)$ "local Kronecker-Weber")

$$\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \cong \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times \cong \widehat{\mathbb{Q}_p^\times} \quad)$$

More gen., $K/\mathbb{Q}_p$ finite

$$\Rightarrow K^{un} = \bigcup_{(m,p)=1} K(\zeta_m),$$

$$\mathrm{Gal}(K^{un}/K) \cong \hat{\mathbb{Z}}$$

k field, k'/k finite sep.

$K = k((t)) \rightsquigarrow G(k') = k'((t))$

Note:

$$G(k^{sep}) = \varinjlim_{\substack{k' \subseteq k^{sep} \\ \text{finite over } k}} k'((t)) \neq k^{sep}((t))$$

(only after completion)

$$\Big(\bigcup_n \mathbb{C}((t^{\frac{1}{n}}))\Big)^\wedge \setminus \bigcup_n \mathbb{C}((t^{\frac{1}{n}}))$$